DIFFUSION AND ABSORPTION OF PARTICLES IN A MEDIUM WITH VARIABLE DENSITY

(DIFFUZIIA I POGLOSHCHENIE CHASTITS V SREDE S PEREMENNOI PLOTNOST'IU)

PMM Vol.28, № 1, 1964, pp.174-177

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(Received April 1, 1963)

Let us consider the problem of diffusion and absorption of particles in the earth's atmosphere. Let $u = u(r,z)$ be the concentration of the diffusing particles. We shall consider the atmospheric density to vary according to the exponential law, and the earth to be plane. In cylindrical coordinates system r,φ,z, whose oz-axis is directed perpendicular to the earth's surface, the diffusion equation

$$u_t = \operatorname{div}\,[D(z)\,\operatorname{grad} u] - \beta(z)\,u$$

has the form

$$u_t = D(z)\,(u_{rr} + r^{-1}n_r + u_{zz}) - \beta(z)\,u + \alpha D(z)\,u_z \qquad (1)$$

Here $D(z)$ is the diffusion coefficient, H is the reduced atmospheric height and $\beta(z)$ is the mean frequency of particle absorption

$$D(z) = D_0 e^{\alpha(z-z_0)}, \qquad \beta(z) = \beta_0 e^{-\alpha(z-z_0)}$$
$$\alpha = 1/H, \qquad D_0 = D(z_0), \qquad \beta_0 = \beta(z_0) \qquad (2)$$

We seek a solution to (1) under the boundary conditions

$$u < \infty \quad \text{as} \quad 0 \leqslant r < \infty, \quad -\infty < z < \infty, \quad u \to 0 \qquad \text{as} \ \sqrt{r^2 + z^2} \to \infty \qquad (3)$$

and initial conditions

$$u = f(r, z) \qquad \text{at} \ t = 0 \qquad (4)$$

where $f(r,z)$ is a given function.

Assuming $u = T(t)\,V(r)\,W(z)$, and substituting into (1), we separate variables

$$T_t + \lambda^2 D_0 T = 0, \qquad U_{rr} + r^{-1}U_r + \varkappa^2 U = 0 \qquad (5)$$

$$W_{zz}^* + \left[\lambda^2 e^{-\alpha(z-z_0)} - \frac{\beta_0}{D_0} e^{-2\alpha(z-z_0)} - \varkappa^2 - \frac{\alpha^2}{4}\right] W^* = 0 \qquad (6)$$

$$\left(\left(W = W^* \exp\left[-\int_{z_0}^{z} \frac{\alpha}{2}\,dz\right]\right) \right)$$

Upon substituting

$$\xi = 2\delta e^{-\alpha(z-z_0)}, \qquad \delta^2 = \frac{\beta_0}{D_0\alpha^2}, \qquad \lambda^* = \frac{\lambda^2}{2\delta\alpha^2}, \qquad \frac{s^2}{4} = \frac{\varkappa^2}{\alpha^2} + \frac{1}{4}$$

Equation (6) reduces to the form

$$W_{\xi\xi}^* + \frac{1}{\xi} W_\xi^* + \left[\lambda^* \frac{1}{\xi} - \frac{1}{4} - \frac{s^2}{4\xi^2}\right] W^* = 0 \quad \left(\lambda^* = \frac{s+1}{2} + n,\ n = 0, 1, 2, \ldots\right) \tag{7}$$

A bounded solution to Equation (7) on $(0,\infty)$ is given by the Laguerre function [1]

$$\omega_n^{(s)}(\xi) = A_n \xi^{-1/2 s} e^{1/2 \xi} \frac{d^n}{d\xi^n} (\xi^{s+n} e^{-\xi})$$

A particular bounded solution to (1) may be given as

$$u = A_n \exp\left[-\int_{z_0}^{z} \frac{\alpha}{2} dz\right] J_0(\varkappa r)\, \omega_n^{(s)}(\xi)\, e^{-\lambda^2 D_0 t}$$

Since Equation (1) is homogeneous and linear, then considering A_n as functions of $\varkappa$, we may write the general solution as a sum of the integrals with respect to $\varkappa$. Integrating with respect to $\varkappa$ from 0 to ∞, and summing over n from 0 to ∞, we get

$$u = \sum_{n=0}^{\infty} \left[\int_0^\infty A_n(\varkappa) \exp\left(-\int_{z_0}^{z} \frac{\alpha}{2} dz\right) J_0(\varkappa r)\, e^{-\lambda^2 D_0 t}\, d\varkappa\right] \omega_n^{(s)}(\xi) \tag{8}$$

In order to determine $A_n(\varkappa)$, we set $t = 0$, and considering condition (4), we have

$$\exp\left(\int_{z_0}^{z} \frac{\alpha}{2} dz\right) f(r, z) = \sum_{n=0}^{\infty} \left[\int_0^\infty A_n(\varkappa) J_0(\varkappa r)\, d\varkappa\right] \omega_n^{(s)}(\xi)$$

The latter expression represents a Laguerre series in the variable ξ. The coefficients of this series equal

$$\int_0^\infty A_n(\varkappa) J_0(\varkappa r)\, d\varkappa = \frac{1}{I_n} \int_0^\infty \exp\left(\int_{z_0}^{z^\circ} \frac{\alpha}{2} dz\right) f(r, z^\circ)\, \omega_n^{(s)}(\xi^\circ)\, d\xi^\circ$$

$$(I_n = n!\, \Gamma(n + s + 1))$$

The resulting expressions for the coefficients of the Laguerre series permit the determination of the quantity $A_n(\varkappa)$, if we use the representation of the given functions as a Fourier-Bessel series

$$A_n(\varkappa) = \frac{\varkappa}{I_n} \int_0^\infty r^\circ J_0(\varkappa r^\circ) \int_0^\infty \exp\left(\int_{z_0}^{z^\circ} \frac{\alpha}{2} dz\right) f(r^\circ, z^\circ)\, \omega_n^{(s)}(\xi^\circ)\, d\xi^\circ dr^\circ$$

Substituting the obtained quantity $A_n(\varkappa)$ in the general solution and interchanging the order of integration and summation, we find

$$u = \int_0^\infty \int_0^\infty f(r^\circ, z^\circ) \exp\left(-\int_{z_0}^{z^\circ} \frac{\alpha}{2} dz\right) \int_0^\infty \sum_{n=1}^{\infty} \frac{1}{I_n} \omega_n^{(s)}(\xi)\, \omega_n^{(s)}(\xi^\circ)\, e^{-\lambda^2 D_0 t} \times$$

$$\times \varkappa J_0(\varkappa r) J_0(\varkappa r^\circ)\, d\varkappa\, r^\circ dr^\circ d\xi^\circ$$

If we observe that

$$\lambda^2 D_0 t = \lambda^* 2\delta\alpha D_0 t = (s + 1 + 2n)\, \alpha t \sqrt{D_0 \beta_0}$$

then upon summing with respect to n , we shall have

$$u = \int_0^\infty \int_0^\infty f(r^\circ, z^\circ) \exp\left(-\int_{z_0}^{z^\circ} \frac{\alpha}{2} dz\right) \frac{1}{2\sinh \alpha t \sqrt{\beta_0 D_0}} \exp\left[-\frac{(\xi+\xi^\circ)}{2} \coth(\alpha t \sqrt{\beta_0 D_0})\right] \times$$

$$\times \int_0^\infty I_s \left[\frac{2\sqrt{\xi\xi^0}}{2\sinh(\alpha t \sqrt{\beta_0 D_0})}\right] \varkappa J_0(\varkappa r) J_0(\varkappa r^\circ)\, d\varkappa r^\circ dr^\circ d\xi^\circ$$

We introduce the variables

$$\xi_1 = \frac{\xi}{2\sinh C}, \qquad \xi_1^\circ = \frac{\xi^\circ}{2\sinh C}, \qquad C = \alpha t \sqrt{\beta_0 D_0}$$

and finally write the solution to (1) as

$$u = \int_0^\infty \int_0^\infty f(r^\circ, z^\circ) \frac{\xi^\circ{}_1^{1/2}}{\xi_1^{1/2}} \exp\left[-(\xi_1+\xi_1^\circ)\cosh C\right] \int_0^\infty I_s(2\sqrt{\xi_1\xi_1^\circ})\, \varkappa J_0(\varkappa r) J_0(\varkappa r^\circ) \times d\varkappa r^\circ dr^\circ d\xi^\circ \tag{9}$$

The concentration of the particles in space may be regarded as the result of the action of instantaneous point particle sources, the distribution of which being given by the function $f(r^\circ, z^\circ)$. Changing the variable of integration in (9) from ξ° to z°, we get

$$u = \int_0^\infty \int_{-\infty}^\infty f(r^\circ, z^\circ)\, G r^\circ dr^\circ dz^\circ.$$

The influence function of the instantaneous point source is

$$G = \frac{\alpha \xi_1^{1/2} \xi_1^{\circ 1/2}}{2\pi} \exp\left[-(\xi_1+\xi_1^\circ)\cosh C\right] \int_0^\infty I_s(2\sqrt{\xi_1\xi_1^\circ})\, \varkappa J_0(\varkappa r^\circ) J_0(\varkappa r)\, d\varkappa \tag{10}$$

Moreover,

$$2\pi \int_0^\infty \int_{-\infty}^\infty f(r^\circ, z^\circ)\, r^\circ dr^\circ dz^\circ = 1$$

To estimate the obtained quantities, and to determine the character of the distribution of the particles in space with increasing time, we use the asymptotic behavior of $I_s(2\sqrt{\xi_1\xi_1^\circ})$. For large values of the argument $2\sqrt{\xi_1\xi_1^\circ}$ the function I_s may be represented as the series

$$I_s(2\sqrt{\xi_1\xi_1^\circ}) = \frac{e^{2\sqrt{\xi_1\xi_1^\circ}}}{(4\pi\sqrt{\xi_1\xi_1^\circ})^{1/2}} \left\{1 + \sum_{k=1}^\infty \frac{(-1)^k (4s^2-1)(4s^2-3^2)\ldots[4s^2-(2k-1)^2]}{k!\, 2^{3k}\, (2\sqrt{\xi_1\xi_1^\circ})^k}\right\}$$

Substituting this into (10), and integrating approximately, we obtain

$$G = \frac{\alpha^3 \xi_1^{1/4} \xi_1^{\circ 1/4}}{8\pi^{3/2}} \exp\left[-(\xi_1+\xi_1^\circ)\cosh C + 2\sqrt{\xi_1\xi_1^\circ} - \frac{1}{(2\xi_1\xi_1^\circ)^{1/2}} - \right.$$

$$\left. - \frac{(r^2+r^{\circ 2})\alpha^2\sqrt{\xi_1\xi_1^\circ}}{4}\right] I_0\left(\frac{r r^\circ \alpha^2 \sqrt{\xi_1\xi_1^\circ}}{2}\right) \tag{11}$$

In the limit case, when $\alpha \to 0$, the medium becomes uniform, while the function G becomes the influence function of the instantaneous point source in a constant density medium, i.e.

$$G = \frac{1}{(2\pi D_0 t)^{3/2}} \exp\left[-\frac{(z-z^\circ)^2 + r^2 + r^{\circ 2}}{4 D_0 t}\right] I_0\left(\frac{r r^\circ}{2 D_0 t}\right)$$

This limiting process verifies the correctness of the assumptions made.

The quantity of absorbed particles equals

$$q = \beta(z)\, G \tag{12}$$

where G is defined by (11), and $\beta(z)$ by (2). If the source acts at the point $\xi_1 = \xi_1^\circ$, $r = r^\circ = 0$ then (12) becomes

$$q = \frac{\alpha^3\beta_0}{\delta}\,\frac{\sinh C}{8\pi^{3/2}}\,\xi_1^{5/4}\xi_1^{\circ 1/4}\exp\left[-(\xi_1+\xi_1^\circ)\cosh C + 2\sqrt{\xi_1\xi_1^\circ} - \frac{1}{4\sqrt{\xi_1\xi_1^\circ}} - \frac{r^2\alpha^2\sqrt{\xi_1\xi_1^\circ}}{2}\right]$$

As an example, let us consider the behavior of extremal points of a cloud, formed as a result of the action of a source at the point with the coordinates $r = 0$, $\xi_1 = \xi_1^\circ$ or $r = 0$, $z = z^\circ$. For this, we equate the derivative

$$\frac{dq}{d\xi_1} = \left(\frac{5}{4\xi_1} - \cosh C + \frac{\xi_1^{\circ 1/2}}{\xi_1^{1/2}} - \frac{1}{8}\,\frac{r^2\alpha^2\xi_1^{\circ 1/2}}{\xi_1^{1/2}} + \frac{1}{8}\,\xi_1^{-3/2}\xi_1^{\circ -1/2}\right) q$$

with zero.

At the points with the coordinates $z = \pm\infty$, the function q vanishes, while the geometric location of the points at which q assumes maximum values, is given by Equation

$$\xi_1\cosh C - \xi_1^{1/2}\xi_1^{\circ 1/2}\left(1 - \frac{1}{8}r^2\alpha^2\right) - \frac{5}{4} + \frac{1}{8}\xi_1^{-1/2}\xi_1^{\circ -1/2} = 0 \tag{13}$$

In this equation we may neglect the last term, then we have a quadratic equation for $\xi_1^{1/2}$, the solution of which gives

$$\xi_1^{1/2} = \xi_1^{\circ 1/2}\,\frac{1}{2\cosh C}\left(1 - {}^1/_8 r^2\alpha^2\right)\left[1 \pm \left(1 + \frac{5\cosh C}{\xi_1^\circ\left(1 - {}^1/_8 r^2\alpha^2\right)^2}\right)^{1/2}\right]$$

We write this expression as

$$\frac{\xi_1^{1/2}}{\xi_1^{\circ 1/2}} = \frac{1}{2\cosh C}\left(1 - {}^1/_8 r^2\alpha^2\right)\left[1 \pm \left(1 + \frac{5\cosh C\sinh C}{\delta\exp\left(-\alpha(z - z^\circ)\right)\left(1 - {}^1/_8 r^2\alpha^2\right)^2}\right)^{1/2}\right] \tag{14}$$

Analysis of (14) shows that when $t \approx 0$,

$$\frac{\xi_1^{1/2}}{\xi_1^{\circ 1/2}} = \begin{cases} 1 & \text{for } {}^1/_8 r^2\alpha^2 = 0 \\ 1 - {}^1/_8 r^2\alpha^2 & \text{for } 0 \leqslant {}^1/_8 r^2\alpha^2 \leqslant 1 \\ 0 & \text{for } 1 \leqslant {}^1/_8 r^2\alpha^2 \end{cases} \tag{15}$$

If

$$\frac{5\cosh C}{\xi_1^\circ\left(1 - {}^1/_8 r^2\alpha^2\right)^2} \gg 1$$

then we get from (14)

$$\frac{\xi_1^{1/4}}{\xi_1^{\circ 1/2}} = \begin{cases} \dfrac{1}{2}\left(\dfrac{5\tanh C\,\alpha\sqrt{D_0\exp\left[\alpha(z^\circ - z_0)\right]}}{\sqrt{\beta_0\exp\left[-\alpha(z^\circ - z_0)\right]}}\right)^{1/2} & \text{for } 0 \leqslant {}^1/_8 r^2\alpha^2 < 1 \quad (16) \\ -\dfrac{5}{4}\,\dfrac{\sinh C\,\alpha\sqrt{D_0\exp\left[\alpha(z^\circ - z_0)\right]}}{\sqrt{\beta_0\exp\left[-\alpha(z^\circ - z_0)\right]}\left(1 - {}^1/_8 r^2\alpha^2\right)} & \text{for } {}^1/_8 r^2\alpha^2 \gg 1 \quad (17) \end{cases}$$

These relations show that the layer with maximum concentration of particles moves in space. The velocity of this displacement decreases with increasing t.

From (16), it follows that the height of the layer with maximum concentration of particles practically does not depend on large values of t. The layer stops. For small values of the quantity $\beta_0\exp\left[-\alpha(z^\circ - z_0)\right]$ it will be located below the line $z = z^\circ$. If $\beta_0\exp\left[-\alpha(z^\circ - z_0)\right]$ has a large value, the layer of maximum particle concentration will be located above the line $z = z^\circ$, or will coincide with it. Expression (17) shows that as the quantity $\frac{1}{8}r^2\alpha^2$ increases, the height of this layer increases.

The author thanks P.I. Kuznetsov for useful adivice.

BIBLIOGRAPHY

Smirnov, V.I., Kurs vysshei matematiki (A course in higher mathematics). Fizmatgiz, (Vol.3, № 7, Part 2), 1958.

Translated by C.K.C.